Particle motion and gravitational lensing in the metric of a dilaton black hole in a de Sitter universe

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We consider the metric exterior to a charged dilaton black hole in a de Sitter universe. We study the motion of a test particle in this metric. Conserved quantities are identified and the Hamilton-Jacobi method is employed for the solutions of the equations of motion. At large distances from the black hole the Hubble expansion of the universe modifies the effective potential such that bound orbits could exist up to an upper limit of the angular momentum per mass for the orbiting test particle. We then study the phenomenon of strong field gravitational lensing by these black holes by extending the standard formalism of strong lensing to the non-asymptotically flat dilaton-de Sitter metric. Expressions for the various lensing quantities are obtained in terms of the metric coefficients.

I. INTRODUCTION

There has been some renewed interest in the study of several different black hole solutions such as the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) dilaton black hole of string theory[1], the brane world black hole with tidal charge[2], the Myers-Perry black hole in the braneworld context[3, 4], and the Einstein-Born-Infeld black hole[5], in recent times. Such studies are expected to shed light on the character of the strong gravitational field and the role of postulated extra dimensions in string and brane theories. Comparison with the standard black hole solutions in four dimensions is important. For example, the braneworld black hole with tidal charge[2] bears formal resemblance to the Reissner-Nordstrom black hole. The question of existence of closed orbits is also interesting, and has been recently investigated[6] in the context of braneworld geometry. In addition, cosmological implications may be significant, as in the case of the Myers-Perry black hole in braneworld cosmology due to its modified evaporation and accretion properties[3, 4].

The study of black holes in the background of a cosmological constant, i.e. in the de Sitter space time, have also attracted some interest for their conjectured relevance in AdS/CFT correspondence[7], and due to the phenomenon of black-hole anti-evaporation [8]. The dilaton black hole solution in a de Sitter universe has been recently derived[9]. The dilaton of the GMGHS black hole is a scalar field occuring in the low energy limit of the string theory. It has an important role on the causal structure and the thermodynamic properties of the black hole. Though it was shown earlier that a dilaton black hole solution in the de Sitter background is not possible with a simple dilaton potential[10], Gao and Zhang[9] circumvented this problem with a dilaton potential being a

combination of Liouville type terms. Such a black hole solution is interesting since it combines stringy features with the backdrop of the present accelerated expansion of the universe.

The motivation for the present work is to explore avenues for possible signatures of cosmological expansion on the orbits of massive bodies and light around dilaton-de Sitter black holes. Gravitational lensing is a widely used tool in modern cosmology. Historically, the theory of gravitational lensing was primarily developed in the weak field thin-lens approximation [11], and some recent studies of weak field lensing in braneworld geometries have also been performed[12]. Although these approximations are sufficiently accurate to discuss any present physical observations, the realization that many galaxies host supermassive central black holes[13] has spurred the development of lensing phenomenology in the strong-field regime[14]. Bozza et al[15] developed an analytical technique for obtaining the deflection angle for the Schwarzschild black hole. Strong field lensing in the Reisser-Nordstrom space-time was then investigated[16]. Bozza[17] extended the analytical theory of strong lensing for a general class of static, spherically symmetric, and asymptotically flat metrics and showed that the logarithmic divergence of the deflection angle at the photon sphere is a common feature of such spacetimes.

From the viewpoint of future observations, these are exciting times in the development of gravitational lensing of compact objects. Newer black hole solutions corresponding to different theoretical models have appeared in the literature[18], and lensing due to several black hole metrics have been analysed recently[12, 14, 17, 19, 20, 21, 22, 23, 24] with the goal of possible observational discrimination in future. Gravitational lensing by the GMGHS black hole has been studied by Bhadra[19]. Subsequently, the strong field phenomenon of retrolensing has been discussed[20], and the position and magnifications of relativistic images due to strong lensing by several braneworld geometries representing the black hole at the centre of our galaxy have been computed[21, 22]. Several interesting features of lensing by rotating black holes

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have also been identified[23]. The theoretical framework of strong field gravitational lensing is being developed further with more suggested applications[25]. In order to detect the existence of relativistic images a high order sensitive instrument with very long baseline interferometry is required. Planned facilities such as Maxim can achieve up to $0.1\mu arcsec$ resolution in the X-ray band[26]. But if future experiments can attain $0.01\mu arcsec$ resolution, and if such relativistic images are detected, then lensing observations will be able to distinguish between the predictions of different black hole models and probe deeply the character of gravity in the strong field regime.

In this paper we present properties of the motion of particles and light in the spacetime of a dilaton-de Sitter black hole. We derive equations of motion and different conserved quantities for particle motion. We also derive the expression for the effective potential which can be used to obtain bound orbits in this geometry. Analysis of the effective potential for the dilaton de-Sitter black hole shows that the Hubble expansion of the universe may significantly contribute to the effective potential only at large distances. In presence of the Hubble term, we derive an upper bound on the value of the angular momentum per mass of the orbiting particle that can be in a bound orbit around the black hole. We then study strong gravitational lensing in this spacetime applying the method of Bozza[17] for the case of a non-asymptotically flat metric for the first time. We obtain the expressions for the lensing observables in terms of the metric coefficients for this geometry.

The paper is organised as follows. The dilaton-de Sitter black hole, its properties, and various limiting cases are briefly described in Section II. In Section III we study several aspects of particle motion in this metric. The effective potential for this metric is obtained and analysed for the existence of bound orbits in the presence of de Sitter expansion of the universe. The investigation of strong field gravitational lensing is performed in the non-asymptotically flat dilaton-de Sitter metric and the expressions for the lensing observables obtained in terms of the metric coefficients in Section IV. We conclude with a summary of our results in Section V.

II. THE DILATON-DE SITTER BLACK HOLE

The action corresponding to gravity coupled to the dilaton and Maxwell fields in four dimensions is given by[9]

$$S = \int d^4x \sqrt{-g} \left[R - 2\partial_\mu \phi \partial^\mu \phi - V(\phi) - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right] \quad (1)$$

where R is the scalar curvature, $F_{\mu\nu}$ is the Maxwell field, ϕ is the dilaton field and $V(\phi)$ is a potential for ϕ. Varying the action with respect to metric, Maxwell field and dilation field, yields respectively, the Einstein, Maxwell and dilaton field equations. Though the dilaton coupling to the electromagnetic sector is non-zero, the Maxwell

equations remain the same as those written in a metric with a conformal factor $e^{-\phi}$. Null geodesics are unaffected by conformal transformations, so that one can continue following null geodesics of the original metric in spite of the coupling between photons and dilatons. The Maxwell equation can be integrated to give, $F_{01} = \frac{Qe^{2\phi}}{f^2}$, where Q is the electric charge

The most general form of any static and spherically symmetric metric is

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)}dr^2 + f(r)^2 d\Omega^2. \quad (2)$$

Using the above form for the metric, the field equations can be reduced to three independent equations[9], i.e.,

$$\frac{1}{f^2}\frac{d}{dr}\left(f^2 U \frac{d\phi}{dr}\right) = \frac{1}{4}\frac{dV}{d\phi} + e^{2\phi}\frac{Q^2}{f^4}, \quad (3)$$

$$\frac{1}{f}\frac{d^2 f}{dr^2} = -\left(\frac{d\phi}{dr}\right)^2, \quad (4)$$

$$\frac{1}{f^2}\frac{d}{dr}\left(2Uf\frac{df}{dr}\right) = \frac{2}{f^2} - V - 2e^{2\phi}\frac{Q^2}{f^4}. \quad (5)$$

For the dilaton-de Sitter black hole[9], f and U are given by

$$f = \sqrt{r(r-2D)}, U = 1 - \frac{2M}{r} - \frac{1}{3}\lambda r(r-2D), \quad (6)$$

where λ is the cosmological constant, and D is the dilaton charge.

Substituting the above expressions into the equations of motion, the dilaton field, dilaton charge and potential become

$$e^{2\phi} = e^{2\phi_0}\left(1 - \frac{2D}{r}\right), \quad (7)$$

$$D = \frac{Q^2 e^{2\phi_0}}{2M}, \quad (8)$$

$$V(\phi) = \frac{4}{3}\lambda + \frac{\lambda}{3}\left[e^{2(\phi-\phi_0)} + e^{-2(\phi-\phi_0)}\right]. \quad (9)$$

where M being the black hole mass, and ϕ_0 the asymptotic constant value of the dilaton. The dilaton potential is the sum of a constant and two Liouville-type terms. In fact, the above structure of the dilaton potential is essential for the existence of dilaton-de Sitter black holes[10].

The action for a dilaton-de Sitter black hole can hence be written as

$$S = \int d^4X \sqrt{-g}(R - 2\partial_\mu \phi \partial^\mu \phi - \frac{4}{3}\lambda$$
$$- \frac{\lambda}{3}\left[e^{2(\phi-\phi_0)} + e^{-2(\phi-\phi_0)}\right]). \quad (10)$$

In terms of the Hubble parameter H, with $H^2 = \frac{\lambda}{3}$, the metric is given by

$$dS^2 = -\left(1 - \frac{2M}{r} - r(r-2D)H^2\right)dt^2 +$$

$$\left(1 - \frac{2M}{r} - r(r-2D)H^2\right)^{-1} dr^2 + r(r-2D)\left(d\Omega^2\right) \quad (11)$$

For $H = 0$ the metric goes to GMGHS black hole[1]. For both $D = 0$ and $H = 0$ it reduces to the well known Schwarzschild metric. Also when $\phi = \phi_0 = 0$ the action of this space-time reduces to the action of a Reisser-Nordstrom-de Sitter black hole. It may be noted here that the solution for a dilaton-de Sitter black hole given by Eq.(11) correspondes to a particular choice of the dilaton potential (9). Other solutions corresponding to other potentials with minima ϕ_0 at which $V(\phi_0) > 0$ are possible, but here we only consider the solution (11) which is written[9] analytically in a closed form.

III. PARTICLE MOTION IN THE DILATON-DE SITTER METRIC

The equations of motion for a test particle of mass m in a curved spacetime with metric $g_{\mu\nu}$ are given by

$$\frac{D^2 X^\mu}{D\tau^2} = 0, \quad (12)$$

where $D/D\tau$ denotes the covariant derivative with respect to proper time τ. The equation can derived from the Lagrangian

$$L = \frac{1}{2}g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu, \quad (13)$$

where an overdot denotes the partial derivatives with respect to an affine parameter λ. For consistency we chose

$$\tau = m\lambda, \quad (14)$$

which is equivalent to

$$g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = -m^2. \quad (15)$$

The conjugate momenta following from the Lagrangian are given by

$$p_\mu = g_{\mu\nu}\dot{X}^\nu, \quad (16)$$

which calculated for the dilaton-de Sitter metric turn out to be

$$
\begin{aligned}
p_r &= \left[1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right]^{-1}\dot{r}, \\
p_t &= -\left[1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right]\dot{t}, \\
p_\theta &= r(r - \frac{Q^2 e^{2\phi_0}}{M})\dot{\theta}, \\
p_\phi &= r(r - \frac{Q^2 e^{2\phi_0}}{M})\sin^2\theta\dot{\phi}.
\end{aligned}
\quad (17)
$$

Since the field is isotropic, one may consider the orbit of the particle confined to the equatorial plane i.e, $\theta = \pi/2$, and $p_\theta = 0$. Then the equations of motion for the dilaton-de Sitter metric are obtained from the variational principle to be

$$\frac{d}{d\lambda}\left[r(r - \frac{Q^2 e^{2\phi_0}}{M})\frac{d\phi}{d\lambda}\right] = 0$$

$$\frac{d}{d\lambda}\left[\left(1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right)\frac{dt}{d\lambda}\right] = 0$$

$$\left[\frac{2M}{r^2} - (2r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right](\frac{dt}{d\lambda})^2$$

$$-\left[1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right]^{-2}$$

$$\times \left[\frac{2M}{r^2} - (2r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right](\frac{dr}{d\lambda})^2$$

$$-2\frac{d}{d\lambda}\left[(1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2)^{-1}\frac{dr}{d\lambda}\right]$$

$$+(2r - \frac{Q^2 e^{2\phi_0}}{M})(\frac{d\phi}{d\lambda})^2 = 0. \quad (18)$$

From the above equations we see that p_t and p_ϕ are constants of motion, but p_r is not. So we get two constants of motion corresponding to the conservation of energy and angular momenta denoted respectively by

$$
\begin{aligned}
p_t &= -E, \\
p_\phi &= \Phi.
\end{aligned}
\quad (19)
$$

In order to solve the system of equations of motion, one employs the Hamilton-Jacobi method. The Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu \quad (20)$$

and

$$
\begin{aligned}
p_\mu &= \frac{\partial S}{\partial x^\mu} \\
p_\nu &= \frac{\partial S}{\partial x^\nu}
\end{aligned}
\quad (21)
$$

and S is the Hamilton-Jacobi action.

The Hamilton-Jacobi equations can be written as

$$-\frac{\partial S}{\partial \lambda} = \mathcal{H} = \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} \quad (22)$$

The action takes the form

$$S = \frac{1}{2}m^2\lambda - Et + S_r + \Phi\phi, \quad (23)$$

where S_r is function of r. From (22) and (23) one obtains

$$(\frac{\partial S}{\partial r})^2(1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2)$$

$$-\frac{E^2}{(1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2)}$$
$$+m^2$$
$$+\frac{\Phi^2}{r(r - \frac{Q^2 e^{2\phi_0}}{M})} = 0 \qquad (24)$$

Equation (24) can be written in compact form:

$$\frac{\partial S_r}{\partial r} = \sigma_r \sqrt{\Re}. \qquad (25)$$

σ_r is the sign function, and $\Re$ can be written as

$$\Re = \frac{\chi}{\Delta}, \qquad (26)$$

where

$$\Delta = (1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2), \qquad (27)$$

and

$$\chi = \frac{E^2}{(1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2)}$$
$$-\frac{\Phi^2}{r(r - \frac{Q^2 e^{2\phi_0}}{M})}$$
$$-m^2. \qquad (28)$$

We can write the Hamilton-Jacobi action in terms of these functions as

$$S = \frac{1}{2}m^2\lambda - Et + \Phi\phi + \sigma_r \int^r \sqrt{\Re}dr. \qquad (29)$$

By differentiating with respect to m, E, and Φ respectively, the solutions of the Hamilton-Jacobi equations can be formally written as

$$\lambda = \int_0^r \frac{dr}{\Delta\sqrt{\Re}}, \qquad (30)$$

$$t = \int_0^r \frac{Edr}{\Delta\sqrt{\Re}}, \qquad (31)$$

$$\phi = \int_0^r \frac{\Phi dr}{\Delta\sqrt{\Re}r(r - \frac{Q^2 e^{2\phi_0}}{M})} \qquad (32)$$

These equations can be expressed in the form of first-order differential equations as

$$\dot{r} = \sigma_r \sqrt{\Re} \left[1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right], \qquad (33)$$

$$\dot{t} = \frac{E}{\left[1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right]}, \qquad (34)$$

$$\dot{\phi} = \frac{\Phi}{r(r - \frac{Q^2 e^{2\phi_0}}{M})}. \qquad (35)$$

A characteristic property of the four-dimensional gravitational field is the existence of bounded orbits located in the exterior of the black hole. One can verify this issue in the context of dilaton-de Sitter gravity by studying the circular orbits ($r = r_0 = $ constant) that are defined by the equations

$$\chi\Delta = 0, \qquad (36)$$

where Δ and χ are given by given by Eqs.(27) and (28) respectively. In order to obtain bound orbits ($E^2 < m^2$) for the metric (11) it is also convenient to use the effective potential.

The equation of the trajectory calculated for the radial motion of a particle with mass m in this metric (11) is

$$(\frac{dr}{dt})^2 = E^2 - V(r) \qquad (37)$$

where the effective potential $V(r)$ for radial motion is given by

$$V(r) = -\left[1 - \frac{2M}{r} - r(r - \frac{Q^2 e^{2\phi_0}}{M})H^2\right] m^2$$
$$\times \left[1 + \frac{\Phi^2}{m^2 r(r - \frac{Q^2 e^{2\phi_0}}{M})}\right]. \qquad (38)$$

$\Phi = r(r - \frac{Q^2 e^{2\phi_0}}{M})\dot{\phi}$ is the ϕ component of the angular momentum. Differentiating $V(r)$ with respect to r, one can find the extrema of the effective potential corresponding to bound orbits, which are given by the roots of the equation $dV/dr = 0$ given by

$$H^2 \left[2r^6 - 10r^5 D + 16r^4 D^2 - 8r^3 D^3\right]$$
$$-2Mr^3 + 8Mr^2 D + 2r^2(\Phi/m)^2 - 6Mr(\Phi/m)^2$$
$$-2rD(\Phi/m)^2 - 8MrD^2 + 8MD(\Phi/m)^2 = 0 \quad (39)$$

For consistency, it can be checked that for $D = 0$, $H = 0$, the above expressions return to their Schwarzschild values. For $H = 0$ one gets the effective potential for the dilaton or GMGHS black hole.

The motion of test particles around a charged dilaton black hole has been studied earlier[27]. Depending on the magnitudes of the charge and angular momentum of the test particle, the radius of the innermost stable orbit shifts compared to that for the Schwarzschild black hole[27]. Here our focus is on the effects of the Hubble expansion on the effective potential for the dilaton-de Sitter black hole. Note first, that the magnitude of the H-dependent terms are negligible compared to the other terms in Eq.(39) at small distances from the black hole. The small value of the Hubble parameter is obviously unable to affect the stability of inner orbits of particles around the black hole. However, as one moves to larger distances from the black hole, the contribution of the

H-dependent terms increase in magnitude. Moreover, far away from the black hole, the dilatonic contribution loses its relevance (D-dependent terms become negligible). Various details of the geodesic structure and particle orbits in Schwarzscild-de Sitter (and anti de Sitter) metrics have been worked out earlier[28]. For the present case, the equation for extrema of the effective potential ($dV/dr = 0$) at large distances can be approximated by

$$H^2 r^5 - M r^2 + (r - 3M)(\Phi/m)^2 \approx 0 \qquad (40)$$

It can be seen from Eq.(40) that the contribution of the Hubble expansion towards the position of particle orbits is effective at distances of $r \sim (M/H^2)^{1/3}$. Further analysis shows that bound orbits exist for the dilaton-de Sitter black hole at these distance scales provided the angular momentum per mass (Φ/m) of the orbiting particle is bounded by the relation

$$\frac{\Phi}{m} \leq \left(\frac{M^2}{H} \right)^{1/3} \qquad (41)$$

Beyond the above limit for Φ/m, the condition for the existence of minima for the effective potential (Eq.(40)) is no longer satisfied, and bound orbits for the dilaton-de Sitter black hole cease to exist at such scales of the order of $r \sim (M/H^2)^{1/3}$ because of the expansion of the universe.

IV. STRONG FIELD LENSING BY DILATON-DE SITTER BLACK HOLES

The dilaton-de Sitter metric in the Schwarzschild coordinate system given by Eq.(11) can be written in the form of a general spherically symmetric and static metric useful for the analysis of gravitation lensing as

$$ds^2 = -A(r)dt^2 + B(r)dx^2 + C(r)\left(d\Omega^2 \right) \qquad (42)$$

where

$$A(r) = \left(1 - \frac{1}{r} - r^2(1 - \frac{\xi}{r})H^2 \right)$$

$$B(r) = \left(1 - \frac{1}{r} - r^2(1 - \frac{\xi}{r})H^2 \right)^{-1}$$

$$C(r) = r^2(1 - \frac{\xi}{r}) \qquad (43)$$

with $r_s = 2M$ as the measure of distance and $\xi = \frac{Q^2 e^{2\phi_0}}{2M^2}$, and H is the Hubble parameter now measured in Schwarzschild radius units. The general formalism of strong field gravitational lensing has been worked out by Bozza[17]. It is required that the equation

$$\frac{C'}{C} = \frac{A'}{A} \qquad (44)$$

admits at least one positive solution the largest of which is defined to be the radius of the photon sphere r_{ps}.

For the dilaton-de Sitter metric we get radius of photon sphere to be $r_{ps} = \frac{3+\xi+\eta}{4}$, where $\eta = \sqrt{9 - 10\xi + \xi^2}$.

Applications of the general framework of strong field lensing[17] have been performed so far on several black hole metrics obtained from string- and brane-inspired models[19, 20, 21, 22]. However, all of the above metrics are asymptotically flat. Here we extend the application of this formalism to the non-asymptotically flat spacetime of the dilaton-de Sitter metric for the first time. A photon emanating from a distant source and having an impact parameter u will approach near the black hole at a minimum distance r_0 before emerging in a different direction (see Figure 1). The impact parameter is related with the closest approach distance by

$$u = \sqrt{\frac{C_0}{A_0}} \qquad (45)$$

where C_0 and A_0 denote the values of the functions evaluted at $r = r_0$. The impact parameter calculated at $r_0 = r_{ps}$ is the minimum impact parameter which for the dilaton-de Sitter metric is given by

$$u_m = \frac{(3+\xi+\eta)\sqrt{3+\eta-3\xi}}{\sqrt{16(3+\xi+\eta) - 64 - H^2(3+\xi+\eta)^3 + 4\xi H^2(3+\xi+\eta)^2}} \qquad (46)$$

It is easy to check that at $H = 0$, and $\xi = 0$, $u_m = \frac{3\sqrt{3}}{2}$, which is the value calculated for Schwarzschild metric, and for $H = 0$, u_m goes to GMGHS black hole value[19].

The deflection angle can be written in terms of the distance of closest approach as

$$\alpha(r_0) = I(r_0) - \pi \qquad (47)$$

where,

$$I(r_0) = \int_{r_0}^{\infty} \frac{2\sqrt{B}}{\sqrt{C}\sqrt{\frac{C}{C_0}\frac{A_0}{A} - 1}} dr. \qquad (48)$$

Substituting the values of the various quantities for the dilaton-de Sitter metric, we get

$$I(r_0) = 2 \int_{r_0}^{\infty} \frac{dr}{r\sqrt{F(r,r_0)}} \qquad (49)$$

where

$$F(r,r_0) = \qquad (50)$$

$$(\frac{r}{r_0})^2 (1 - \frac{\xi}{r})^2 (1 - \frac{\xi}{r_0})^{-1} (1 - \frac{1}{r_0}) - (1 - \frac{\xi}{r})(1 - \frac{1}{r}))$$

Puttting $z = 1 - \frac{r_0}{r}$, one gets

$$I(r_0) = \int_0^1 R(z,r_0) f(z,r_0) dz. \qquad (51)$$

where

$$R = 2\frac{\sqrt{(1 - \frac{\xi}{r_0})}}{1 - \frac{\xi}{r_0} + z\frac{\xi}{r_0}}, \tag{52}$$

is regular for all values of z, and

$$f(z, r_0) = [(1 - \frac{1}{r_0} - (1 - \frac{1-z}{r_0})$$
$$\times (1 - z)^2 (1 - \frac{\xi}{r_0})(1 - \frac{1-z}{r_0}\xi)^{-1}]^{-1/2} \tag{53}$$

diverges for $z \to 0$. We expand the argument of the squareroot in $f(z, r_0)$ to the second order in z and get,

$$f(z, r_0) \sim f_0(z, r_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}} \tag{54}$$

with

$$\alpha = -\frac{3}{r_0} + 2$$
$$+ \frac{\xi}{r_0}(1 - \frac{\xi}{r_0})^{-1}(1 - \frac{1}{r_0}) \tag{55}$$

and

$$\beta = \frac{1}{2}[\frac{6}{r_0} - 2 + \frac{\xi}{r_0}(1 - \frac{\xi}{r_0})^{-1}(\frac{6}{r_0} - 4)$$
$$- 2(\frac{\xi}{r_0})^2(1 - \frac{\xi}{r_0})^{-2}(1 - \frac{1}{r_0})] \tag{56}$$

Again, α and β reduce to their Schwarzschild values[17] for $\xi = 0$, and they are the same for the GMGHS metric[19] since the H-dependence has already dropped out.

The total deflection angle is given by $I(r_0) = I_D(r_0) + I_R(r_0)$, where

$$I_D(r_0) = \int_0^1 R(0, r_{ps}) f_0(z, r_0) dz, \tag{57}$$

includes the divergent part with $R(0, r_{ps}) = \frac{2}{\sqrt{1 - \frac{\xi}{r_{ps}}}}$.
The second term in the deflection angle given by

$$I_R(r_0) = \int_0^1 g(z, r_0) dz$$
$$= \int_0^1 [R(z, r_0) f(z, r_0) - R(0, r_{ps}) f_0(z, r_0)] dz \tag{58}$$

is the original integral with the divergence substracted. One can expand $I_R(r_0)$ in power of $(r_0 - r_{ps})$ and considering the first expansion term we get

$$I_R(r_0) = \int_0^1 g(z, r_{ps}) dz + O(r_0 - r_{ps}) \tag{59}$$

This integral can be evaluated and the regular term of the deflection angle for the dilaton-de Sitter metric is obtained to be

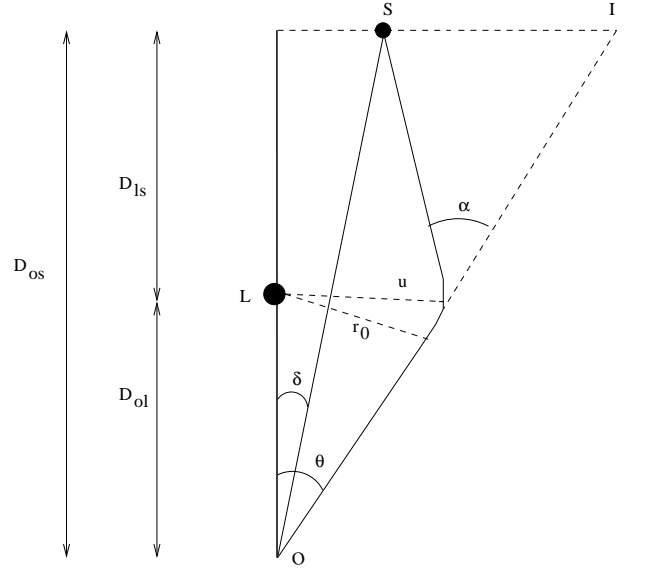$$I_R(r_{ps}) = \frac{2}{\sqrt{1 - \frac{\xi}{r_{ps}}}\sqrt{\beta}}[ln(4\sqrt{\beta})$$

FIG. 1: Gravitatioanal lensing for point like mass object M. A light ray from the source S passes the lens with an impact parameter u, and is deflected by an angle α. The observer sees an image I of the source at the angular position θ.

$$- ln(\frac{2\beta + \gamma}{\sqrt{\beta}} + 2\sqrt{\beta + \gamma + \delta})] \tag{60}$$

where the expressions for γ and δ are given by,

$$\gamma = \frac{r_{ps} - 4\xi + \xi r_{ps}}{r_{ps}(\xi - r_{ps})} \tag{61}$$

$$\delta = \frac{\xi}{r_{ps}(\xi - r_{ps})} \tag{62}$$

All the coefficients are evaluated at the point $r_0 = r_{ps}$, and we get only the regular part of the deflection angle[17]. Setting $\xi = 0$ one obtains all the coefficients for the Schwarzschild metric.

The expression for the strong field limit of the deflection angle may be written as[17]

$$\alpha_\theta = -\bar{a} ln[\frac{\theta D_{ol}}{u_{rps}} - 1] + \bar{b}, \tag{63}$$

where,

$$\bar{a} = \frac{R(0, r_{ps})}{2\sqrt{\beta_{rps}}} \tag{64}$$

$$R(0, r_{ps}) = 2\frac{\sqrt{3 + \eta + \xi}}{\sqrt{3 + \eta - 3\xi}} \tag{65}$$

$$\bar{b} = -\pi + b_R + \bar{a} ln\left[\frac{2\beta_{rps}}{1 - \frac{1}{r_{ps}} - r_{ps}^2(1 - \frac{\xi}{r_{ps}})H^2}\right] \tag{66}$$

$$b_R = \frac{R(0, r_{ps})}{2\sqrt{\beta_{r_{ps}}}}[2ln(4\sqrt{\beta_{r_{ps}}})$$

$$-ln(\frac{2\beta_{r_{ps}} + \gamma_{r_{ps}}}{\sqrt{\beta_{r_{ps}}}} + 2\sqrt{\beta_{r_{ps}} + \gamma_{r_{ps}} + \delta_{r_{ps}}})] \qquad (67)$$

Note that the expression for $\bar{b}$ given in Eq.(66) contains a H-dependent term which introduces a small correction to the lensing angle.

In strong gravitational lensing there may exist n relativistic images given by the number of times a light ray loops around the black hole. The positions of the source and the images are related through the lens equation derived by Virbhadra and Ellis[14] given by

$$tan\delta = tan\theta - \frac{D_{ds}}{D_s}[tan\theta + tan(\alpha - \theta)] \qquad (68)$$

The magnification μ_n of the n-th relativistic image is given by[17]

$$\mu_n = \frac{1}{(\delta/\theta)\partial\delta\partial\theta}|_{\theta_n} \simeq \frac{u_m^2 e_n(1 + e_n)D_s}{\bar{a}\delta D_{ds}D_d^2} \qquad (69)$$

where $e_n = e^{(\bar{b} - 2n\pi)/\bar{a}}$. The expressions for the various lensing observables can be obtained in terms of the metric parameters. For $n \to \infty$ an observable θ_∞ can be defined[17] representing the asymptotic position approached by a set of images. The minimum impact parameter can then be obtained as

$$u_m = D_d\theta_\infty \qquad (70)$$

In the simplest situation where only the outermost image θ_1 is resolved as a single image, while all the remaining ones are packed together at θ_∞, two lensing observables can be defined as[17]

$$S = \theta_1 - \theta_\infty \qquad (71)$$

representing the separation between the first image and the others, and

$$\mathcal{R} = \frac{\mu_1}{\sum\limits_{n=2}^{\infty} \mu_n} \qquad (72)$$

corresponding to the ratio between the flux of the first image and the flux coming from all the other images.

In terms of the deflection angle parameters $\bar{a}$ and $\bar{b}$, these observables can be written as[17]

$$S = \theta_\infty e^{\bar{b}/\bar{a} - 2\pi/\bar{a}} \qquad (73)$$

$$\mathcal{R} = e^{2\pi/\bar{a}} \qquad (74)$$

The above equations (73) and (74) can be inverted to express $\bar{a}$ and $\bar{b}$ in terms of the image separation S and the flux ratio $\mathcal{R}$. Therefore the knowledge of these two observables can be used to reconstruct the deflection angle

given by Eq.(63). The aim of strong field gravitational lensing is to detect the relativistic images corresponding to specific lensing candidates and measure their separations and flux ratios. Once this is accomplished, the observed data could be compared with the theoretical coefficients obtained using various metrics. A precise set of observational data for strong gravitational lensing, if obtained, could therefore be able discriminate between different models of gravity.

Lensing observables such as the angular position of the relativistic images (θ_∞), the angular separation of the outermost relativistic image with the remaining bunch of relativistic images ($\mathcal{S}$), and the relativistic magnification of the outermost relativistic image with respect to the other relativistic images (r_m) have been computed earlier for the galactic centre black hole considering various candidate geometries, viz. the Schwarzschild black hole[17], the braneworld black hole with negative tidal charge[2, 21, 22], and the Reissner-Nordstrom black hole (which can also be considered as a GMGHS black hole[1, 19] with dilaton potential minimum $\phi_0 = 0$). All these metrics have aroused interest recently as black hole candidates representing gravity modified in separate contexts. It should be mentioned here that according to recent observational results[29], supermassive black holes including the one in our galactic centre seem to be spinning at considerable rates as described by the Kerr solution, but there is lack of evidence regarding any charged astrophysical black holes. From the theoretical viewpoint, no black hole solution has yet been found which incorporates both the dilaton and de Sitter expansion for the Kerr black hole, though there exit some interesting applications of particle motion in the Kerr-de Sitter metric[30]. Further, strong gravitational lensing by the Kerr metric itself is endowed with additional features, and the full phenomenological implications are in the process of being worked out[23].

Considering the supermassive black hole in the galactic center (with its mass $4.3 \times 10^6 M_\odot$ and its distance from us $8kpc$[31]) as a charged dilaton black hole in a cosmological background, one can estimate the observables of strong lensing. However, due to the small value of the Hubble parameter ($H = 7.7 \times 10^{-27}$s) the de Sitter expansion makes a rather negligible contribution to the actual values of the lensing observables. This is apparent from considering the minimum impact parameter given by Eq.(46), which can be expanded in terms of powers of H as

$$u_m = (u_m)_{GMGHS}\left(1 + \frac{(3 + \xi + \eta)(3 - 3\xi + \eta)^{1/2}H^2}{2[16(3 + \xi + \eta) - 64]}\right)$$
$$+ O(H^4) \qquad (75)$$

where $(u_m)_{GMGHS}$ denotes the minimum impact parameter for the GMGHS metric. Hence one sees that the Hubble expansion makes a rather tiny modification ($O(H^2)$) to the observable values, that are too insignificant to enable, for example, the dilaton-de Sitter metric to be distinguished from the GMGHS metric.

V. CONCLUSIONS

In this paper we have discussed several features of the motion of particles and light in the spacetime of a charged dilaton black hole in a cosmological background. The black hole solution recently obtained[9] corresponding to the dilaton-de Sitter metric could be relevant for the study of string theoretic implications in the context of the presently accelerating universe. The conserved quantities in the dilaton-de Sitter metric have been identified and the Hamilton-Jacobi method employed for the equations of motion. We have also obtained the effective potential for this metric in order to analyse the existence of bound orbits in this metric in an expanding universe. Close to the black hole the dilatonic contribution in the effective potential dominates over that of the Hubble term which is rather negligible. However, we find that at large distances from the black hole the effective potential could be significantly affected by the Hubble expansion. We have derived an expression for the upper value of the angular momentum per mass of the orbiting particle up to which bound orbits could exist in this metric.

We have further investigated strong field gravitational lensing in the dilaton-de Sitter metric. Gravitational lensing promises to be a powerfull tool in future observations for probing the strong field character of gravity that may be able to incorporate signatures of extra dimensions as in string or brane models. Here we have extended the application of the standard strong lensing framework[17] used previously for only asymptotically flat spacetimes, for obtaining the expressions for the different lensing variables as functions of the metric parameters of the non-asymptotically flat dilaton-de Sitter metric. We note however, that the value of the Hubble parameter is too small to impact the position of relativistic images and other strong lensing quantities in any observationally significant way. We conclude with the rider that though our present analysis may not provide accurate estimates of the actual position of stable orbits or of lensing observables for any astrophysical black hole in the expanding universe, these calculations may serve to motivate future analysis using more advanced theoretical techniques suitable to handle the realistic situations including effects such as black hole rotation.

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